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Individual differences in students' knowing and learning about fractions: Evidence from an in-depth qualitative study

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Abstract

We present the results of an in-depth qualitative study that examined ninth graders' conceptual and procedural knowledge of fractions as well as their approach to mathematics learning, in particular fraction learning. We traced individual differences, even extreme, in the way that students combine the two kinds of knowledge. We also provide preliminary evidence indicating that students with strong conceptual fraction knowledge adopt a deep approach to mathematics learning (associated with the intention to understand), whereas students with poor conceptual fraction knowledge adopt a superficial approach (associated with the intention to reproduce). These findings suggest that students differ in the way they reason and learn about fractions in systematic ways and could be used to inform future quantitative studies.

Keywords: fractions; conceptual/procedural knowledge; individual differences; learning approach

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1. Theoretical background

The distinction between procedural and conceptual knowledge has elicited considerable research and discussion among researchers in the fields of cognitive-developmental psychology and mathematics education. Procedural knowledge is defined as the ability to execute action sequences to solve problems and is usually tied to specific problem types, whereas conceptual knowledge is defined as knowledge of concepts pertaining to a domain and related principles (Rittle-Johnson and Schneider, in press).

The relation between the two types of knowledge, particularly with respect to their order of acquisition has elicited considerable discussion, and there is evidence in favour of contradictory views – in the words of Rittle-Johnson, Siegler, and Alibali (2001), "concepts-first" and "procedures-first" theories. According to concepts-first theories, children develop (or are born with) conceptual knowledge in a domain and then use this knowledge to select procedures for solving problems. According to procedures-first theories, children learn procedures for solving problems in a domain and later extract domain concepts from repeated experience in solving problems. In the area of mathematics education research, the two types of knowledge (sometimes referred to by other terms) are deemed practically inseparable (Gilmore & Papadatou-Pastou, 2009; Hiebert & Wearne, 1996). Nevertheless, it is assumed that procedural knowledge plays an important role in the development of conceptual understanding (Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991). More specifically, it is suggested that mathematical concepts develop out of related mathematical processes.

Such accounts share two common background assumptions, namely that there is a single developmental path and that this path is independent of the particular domain considered. Rittle-Johnson and Siegler (1998) challenged the latter providing evidence that the order of acquisition many vary, depending of the domain considered. In any case, the two types of knowledge appear closely related. Thus, Rittle-Johnson et al. (2001) argued for an iterative model, according to which the two types of knowledge develop in a hand-over-hand process and gains in one type of knowledge lead to improvements in the other. This model is supported by empirical evidence and seems to provide an adequate description of the relation between conceptual and procedural knowledge (Rittle-Johnson & Schneider, in press). Nevertheless, there is evidence that sometimes the development of one type of knowledge does not necessarily lead to the development of the other. Indeed, in the area of fraction learning it has been shown that some students have the ability to perform fraction procedures without exhibiting comparable conceptual understanding or without being able to explain why they are using these procedures (Kerslake, 1986; Peck & Jencks, 1981). On the other hand, Resnick (1982) presented evidence showing that some children may exhibit conceptual understanding of principles underlying subtraction without showing procedural fluency.

Recently, a different explanation for the contradictory findings has been proposed, namely that not enough attention has been paid to the individual differences in the way that students combine the two types of knowledge (Gilmore & Bryant, 2008; Gilmore & Papadatou-Pastou, 2009; Hallett, Bryant, & Nunes, 2010; Hallett, Nunes, Bryant and Thrope, 2012). Hallett and colleagues examined the procedural and conceptual fraction knowledge of students at Grade 4 and 5 (2010) as well as at Grade 6 and 8 (2012). They identified groups of students who had strong (or weak) procedural as well as conceptual knowledge. However, they also consistently traced two substantial groups of students who demonstrated relative strength with one form of knowledge and weakness with the other, with differences between the two types of knowledge becoming less salient with age. These findings challenge the assumption that all children follow a uniform sequence in gaining the two types of knowledge (see also Canobi, Reeve, & Pattison, 2003).

In their attempts to explain how such individual differences arise, some researchers appealed to differences in students' prior knowledge in the domain in question (Schneider, Rittle Johnson, & Star, 2011); differences in students' cognitive profiles (Gilmore & Bryant, 2008; Hallett et al., 2012) and differences in students' educational experiences (Canobi, 2004; Gilmore & Bryant, 2008; Hallett et al., 2012). However, empirical evidence in support of these assumptions is so far lacking. For example, Schneider et al. (2011) found no evidence supporting the hypothesis that the correlation between the two kinds of knowledge might vary with different levels of prior knowledge in the area of equation solving. Hallett et al. (2012) investigated whether individual differences in procedural and conceptual knowledge of fractions can be explained by differences in students' general procedural and conceptual ability (measured by standardized tests); they found no such evidence. In addition, Hallett et al. (2012) examined the role of school experience, which they measured as school attendance, that is, they investigated whether attending different schools could explain the individual differences in question; they found no such relation.

Further research, possibly with different measures, is necessary to clarify the role of the above factors in individual differences in procedural and conceptual knowledge, in particular of fractions. We argue that a factor also worth investigating is the individual student's learning approach to mathematics.

In the literature there is an overarching distinction between the deep approach to learning, associated with the individual's intention to understand; and the surface approach, associated with the individual's intention to reproduce. There are several ways of characterizing each learning approach, mainly adapted to tertiary education (Entwistle & McCune, 2004). Stathopoulou and Vosniadou (2007) proposed a model, which was tested with secondary students. They included three categories for each learning approach, namely Goals, (study) Strategy Use, and Awareness of Understanding. A deep approach to learning involves goals of personal making of meaning, deep study strategy use (e.g., integration of ideas), and high awareness of understanding. A superficial approach involves performance goals, superficial strategy use (e.g., rote learning), and low awareness of understanding. Using these categories, Stathopoulou and Vosniadou showed that students with strong conceptual understanding of science concepts adopted a deep approach to science learning, whereas students with poor conceptual understanding adopted a superficial approach. A similar association might be present in the case of mathematics as well. Indeed, a student that follows a deep learning approach to mathematics is more likely to pay attention to the concepts and principles in the domain in question, to be aware of conceptual difficulties, and to invest the effort necessary to overcome them. On the contrary, a student with a superficial approach is more likely to focus on memorizing procedures, especially if procedures are emphasized in instruction, as is often the case (Moss, 2005).

Before we formulate our hypotheses, we turn to a methodological issue, namely the difficulty to measure the two types of knowledge validly and independently of each other (e.g., Gilmore & Bryant, 2006; Hiebert & Wearne, 1996; Rittle-Johnson & Schneider, in press; Schneider & Stern, 2010; Silver, 1986). The development of a procedural test that would be conceptual free (and vice versa) is a challenging task, since this type of tests may be person, content and context sensitive (Haapasalo & Kadijevich, 2000; Schneider et al., 2011). Moreover, for tasks administered in paper-and-pencil tests, it is often impossible to decide how the student actually solved the task. For such reasons, Hiebert and Wearne (1996) suggested that attention should be also paid to students' solution strategies (see also Faulkenberry, 2013). A distinction between procedural and conceptual strategies (Alsawaie, 2011; Clarke & Roche, 2009; Yang, Reys, & Reys, 2007) is relevant at this point: Procedural strategies are related to rules and exact computation algorithms learnt from instruction. Conceptual strategies, on the other hand, are diverse, and tailored to the specific problem at hand; they are mostly invented by (some) students themselves that use them flexibly in order to avoid lengthy computations as well as to deal with unfamiliar problems (see also Smith, 1995).

In this study, we examined ninth graders' conceptual and procedural fraction knowledge. Taking into account the methodological issue mentioned above, we designed a qualitative study in order to also monitor students' strategies. Similarly to Hallett et al. (2010, 2012), we hypothesized that there are individual differences in the way students combine the two kinds of knowledge. We were particularly interested in extreme cases, namely students with strong conceptual knowledge and weak procedural knowledge, and vice versa. Such cases are theoretically interesting, since they are not compatible with the iterative model (Rittle-Johnson et al., 2001). Moreover, tracing extreme cases at grade 9 would indicate that individual differences may persist, although the general tendency is for them to become less salient with age (Hallett et al., 2012).

In addition, we examined students' learning approach to mathematics learning, particularly fraction learning. Following Stathopoulou and Vosniadou (2007), we explored whether students with strong conceptual knowledge adopt a deep learning approach to mathematics, whereas students with weak conceptual knowledge adopt a superficial approach.

2. Methodology

2.1 Participants

The participants were seven Greek students at grade nine (three girls), from seven different schools in the area of Athens. The selection of the participants was not random. First, based on their school grades, all participants could be characterized as medium level students in mathematics. Second, they all had the same mathematics tutor, starting from the last grades of the elementary school. Their tutor provided information about their mathematical behaviour. Based on this information, we had reasons to expect some variation in their conceptual and procedural knowledge of fractions.

We note that by grade seven Greek students are taught all the material related to fractions as well as decimals, and are introduced to the term “rational numbers”. We stress that at the moment this study took place the mathematics curriculum as well as the mathematics textbooks, were “traditional’, in the sense that they emphasized general, computation-intensive procedures for dealing with fraction tasks (Smith, 1995). Consider, for example, that mental calculations and estimation strategies were not among the curricular goals. Based on information provided by our participants’ tutor, who had extensive knowledge about their homework assignments as well as their assessment tests on a long-term basis, we had good reasons to believe that instruction relied heavily on the textbooks, at least with respect to what students were expected to do.

2.2 Materials

We used thirty fraction tasks grouped in four categories (see Appendix A). Category A included five procedural tasks, that is, tasks that for which a standard procedure was taught at school: four tasks that examined operations with fractions (Q.1.1-Q.1.4); and one task that required conversion to an equivalent fraction (Q.1.5).

Category B, consisting of eight tasks, targeted on conceptual knowledge. Four tasks involved fraction representations (Q.1.6-Q.1.9); one task required recognizing fraction as a ratio (Q.1.10); one item focused in the role of the unit of reference (Q.1.11); and two tasks targeted on the understanding of the effect of multiplication and division with fractions (Q.1.12, Q.1.13). There were no tasks similar to Q.1.10-Q.1.13 in the textbooks, either at the elementary, or at the secondary level. On the other hand, the area model for the representation of fractions was salient in the elementary school textbooks, but unlike Q.1.8., the shape was typically given, already equally partitioned; examples of improper fraction representations were scarce (Q.1.9), and there was no task similar to Q.1.7.

Category C consisted of seven comparison (Q.1.14-Q.1.17, Q.1.20-Q.1.22) and two ordering tasks (Q.1.18, Q-1.19). Although these tasks could be solved by standard methods taught at school, they could also be solved by a variety of conceptual strategies.

Finally, the tasks of the Category D required deep conceptual understanding or the combination of conceptual understanding and procedural fluency. More specifically, there were two tasks regarding locating fractions on the number-line (Q.1.23, Q.1.24); one problem that involved an intensive quantity and required the comparison of ratios (Q.1.25); one task regarding estimation of a fraction sum (Q.1.26); one task that required substituting variables with non-natural numbers (Q.1.27); one task that tested the use of the inverse relationship between addition and subtraction, as well as between multiplication and division with fractions

(Q.1.28); and two tasks targeting the dense ordering of rational numbers (Q.1.29, Q.1.30). There were no tasks similar to Q.1.27-Q.1.30 in the mathematics textbooks. Locating fractions on the number line was presented at the secondary level (Grade 7), albeit not particularly emphasized.

The selection and categorization of the tasks was based on relevant literature (e.g., Clarke & Roche, 2009; Hallett et al., 2010, 2012; McIntosh, Reys, & Reys, 1992; Moss & Case, 1999; Smith, 1995). We note that we included items targeting students' awareness of the differences between natural and rational numbers (e.g., Q.1.12, Q.1.13, Q.1.29, Q.1.30) which is considered an important aspect of conceptual knowledge (Vamvakoussi & Vosniadou, 2010; McMullen, Laakkonen, Hannula-Sormumen, & Lehtinen, 2014). We also used a considerable number of tasks related to fraction magnitude (e.g., Category D tasks, Q.1.23, Q.1.24) (for the importance of accessing fraction magnitude in students' developing knowledge see Siegler & Pyke, 2013). We stress, however, that this categorization was tentative, since we also looked into students' strategies. This consideration is particularly important for Category C tasks, but relevant for all tasks.

In addition, we developed twelve items so as to explore students' learning approach (deep/superficial) to fraction and, more generally, to mathematics learning (see Appendix B). The items were presented as scenarios describing a situation that the student had to react to.

2.3 Procedure

In the first phase of the study each student was asked to solve the fraction tasks, thinking aloud and explaining their answers. No time limit was imposed. In the second phase three participants were selected to participate in an in-depth, semi-structured individual interview about their learning approach to mathematics. Because this was a first attempt to explore a potential relation between individual differences in conceptual and procedural fraction knowledge and the individual's learning approach, we selected one student with strong procedural, but weak conceptual knowledge; one student with strong conceptual but weak procedural knowledge; and one student who combined both procedural and conceptual knowledge. These students were additionally asked to comment on the responses of the first questionnaire (certainty about the solution, awareness of their performance in the tasks). The second interview took place about one week later. Each interview lasted about one hour. All interviews were recorded and transcribed.

2.4 Data Analysis

First, we assessed the accuracy of students' responses in all tasks. Second, we examined the strategies used. We categorized a strategy as procedural, if it was based on instructed rules and procedures related to our research tasks. Based on mathematics textbooks, as well as information by our participants' mathematics tutor, we categorized as procedural strategies the standard algorithms for fraction operations; and transformation strategies (Smith, 1995), namely converting to equivalent fractions, similar fractions, decimals, or mixed numbers. Transformation strategies are relevant to operations as well as comparison, and they were over-emphasized in the textbooks. We also categorized as procedural the instructed method for Q.1.25, namely the construction of a 2x2 table placing the like quantities one below the other, and forming and comparing the ratios. Regarding the placement of fractions on the number line, the instructed method involved segmenting the unit in the appropriate number of parts. Finally, given the salience of the area model for the representation of fractions, particularly in the elementary grades, we reasoned that it had the status of definition for fractions. We thus did not consider that students used a strategy, either conceptual, or procedural in the related tasks (Q.1.6–Q.1.9).

We categorized as conceptual the strategies that were not based on instructed procedures. For comparison tasks, such strategies involved, for example, the use or reference numbers, such as the unit and one half; and also residual thinking, that is, comparing the complementary fractions (Alsawaie, 2011; Clarke & Roche, 2009; Smith, 2005; Yang et al., 2007). In a more general fashion, we categorized as conceptual strategies the ones that relied on estimation of fraction magnitudes, on spontaneous use of representations, and on spotting and employing the multiplicative relations present in the task at hand (e.g., in Q.1.25).

We categorized a strategy as conceptual/procedural if it involved conceptual and procedural features, such as adjusting a procedural strategy to deal with a novel task. A prominent example was the use of a transformation strategy, namely converting to equivalent fractions, as a first step to deal with Q.1.30, combined with the idea that this process can be repeated infinitely many times.

We also note that in certain cases students provided immediate responses that were not based on a specific strategy; rather, they relied on a holistic understanding of the situation at hand. This was the case mainly for tasks targeting the differences between natural numbers and fractions (Q.1.12, Q.1.13, Q.1.29, Q.1.30). For example, some students answered immediately that there is no other number between 2/5 and 3/5, directly drawing on their natural number knowledge. We categorized the strategy of relying on natural number knowledge as conceptual.

For the second phase of the study, the categories (i.e., Goals, Strategy Use, and Awareness of Understanding) and the related indicators used by Stathopoulou and Vosniadou (2007) were our starting point for the analysis. We reviewed all transcripts and coded them when possible. We selected sentences as unit of analysis, but in some cases we used paragraphs so as to obtain a sense of the whole. We looked for utterances that included keywords pertaining to the indicators of each category (e.g., remember, memorize, memory and similar expressions for the indicator ''rote-learning'' as a superficial Strategy Use). We placed the sentences in the coding categories according to the initial indicators and developed new indicators when needed. After coding, data that could not be coded were identified and analyzed to determine if they represented a new category. One new category emerged, namely Engagement Factors, consisting of two sub-categories: Preferred Tasks/Strategies (conceptual/procedural), and also Motivation (intellectual challenge/coping). In addition, we replaced the category Awareness of Understanding with the more general category Awareness with indicators pertaining to awareness of understanding (high/low) as well as to awareness of the effectiveness of one's personal study strategies (high/low). The categories are presented in Table 5.

3. Results of the 1st phase of the study

Tables 1-4 present how students performed in the tasks of Categories A-D, respectively; and the type of strategy (conceptual, procedural, or a combination of both) they used in each task.

As shown in Tables 1-4, students 1, 2, and 3 were rather successful across all task categories. Students 4, 5, and 6 were successful in Categories A and C, but not in Categories B and D. Student 7 failed in Category A, but was rather successful in Categories B, C, D. We placed the students in three profiles: a) Conceptual-Procedural (Students 1, 2, and 3); b) Procedural (Students 4, 5, and 6); and c) Conceptual (Student 7). In the following we present these profiles in more detail.

3.1 Conceptual - Procedural Profile

The conceptual-procedural students succeeded in all tasks of Category A using procedural strategies, that is, standard algorithms (Table 1). Student 1 and Student 3 (hereafter, Kosmas) also succeeded in all tasks of Category B (Table 2). All three students relied heavily on conceptual strategies (reference numbers, residual thinking) to deal with the tasks of Category C (Table 3). All three performed well in the tasks of this category, with Kosmas responding correctly to all tasks.

Table 1

Students' Performance (Success, Failure) and Type of Strategy Used (Conceptual, Procedural, Or Conceptual-Procedural) in the Tasks of Category A

Student	Q.1.1	Q.1.2	Q.1.3	Q.1.4	Q.1.5	Profile
1	S, P	S, P	S, P	S, P	S, P	Conceptual/Procedural
2	S, P	S, P	S, P	S, P	S, P	
3 (Kosmas)	S, P	S, P	S, P	S, P	S, P	
4	S, P	S, P	S, P	S, P	F, P	Procedural
5	S, P	S, P	S, P	S, P	S, P	
6 (Stella)	S, P	S, P	S, P	S, P	S, P	
7 (Filio)	F, P	F, P	F, P	F, P	S, P	Conceptual

Kosmas was the only student who responded correctly to all tasks of Category 4. In general, however, all three students performed well in Category D tasks, showing a rather strong conceptual understanding, combined with procedural fluency. A good indicator of their conceptual understanding is their responses to the density tasks (Q.1.29, Q.1.30), in particular to the first that is the most challenging. Student 2 and Kosmas provided an impressively sophisticated answer, stating explicitly that there is no such number and explained that, given any number, no matter how small, one can always find a smaller one. Student 1, on the other hand, assumed that such a number exists, thus typically his answer is incorrect; however, he stated that this number cannot be found, not even by a computer; and described it as "zero point zero, followed by infinitely many zeroes, and one unit in the end".

These students' tendency to prefer conceptual over procedural strategies manifested itself in the tasks of Category D as well. None of them applied the instructed method to solve Q.1.25; instead, they focused on the relations between the quantities involved. In the words of Student 2: "Stella's milk tastes sweeter, because George dissolved the double quantity of chocolate in the triple quantity of milk".

The data presented in Tables 1-4 show that Kosmas was the only one who succeeded in all tasks. Moreover, Kosmas's responses were more elaborated than his peers' in terms of completeness as well as of the explanations he provided. Consider, for example, Q.1.26 that asked for the estimation of 7/15 and 5/12. All three students noticed that each addend was smaller than 1/2 and concluded that the sum was smaller than the unit. Kosmas, however, went farther to notice that "This sum equals the unit minus 0.5/15+1/12. The missing part is close to 0.1; more precisely, a bit bigger than 0.1". He reached this close estimate of the missing part mainly via mental calculations, writing down some of the intermediate results.

Table 2

Students' Performance (Success, Failure) and Type of Strategy Used (Conceptual, Procedural, Or Conceptual-Procedural) in the Tasks of Category B

Student	Q.1.6	Q.1.7	Q.1.8	Q.1.9	Q.1.10	Q.1.11	Q.1.12	Q.1.13	Profile
1	S	S	S	S	S	S	S, C	S, C	Conceptual/Procedural
2	S	S	S	S	S	F	S, C	S, C	
3 (Kosmas)	S	S	S	S	S	S	S, C	S, C	
4	S	F	F	F	F	F	F, C	F, C	Procedural
5	S	F	F	F	F	F	F, C	F, C	
6 (Stella)	F	F	F	F	F	F	F, C	F, C	
7 (Filio)	S	S	S	S	S	S	S, C	S, C	Conceptual

Table 3

Students' Performance (Success, Failure) and Type of Strategy Used (Conceptual, Procedural, Or Conceptual-Procedural) in the Tasks of Category C

Student	Q.1.14	Q.1.15	Q.1.16	Q.1.17	Q.1.18	Q.1.19	Q.1.20	Q.1.21	Q.1.22	Profile
1	S, C	S, C	S, C	S, C	S, C	S, C	F, C	S, C	S, C	Conceptual/ Procedural
2	S, C	S, C	S, C	S, C	S, C	F, C/P	S, C	S, C	S, C	
3 (Kosmas)	S, C	S, C	S, C	S, C	S, C	S, C	S, C	S, C	S, C	
4	S, P	S, P	S, P	S, P	S, P	S, P	S, P	S, P	S, P	Procedural
5	S, P	S, P	S, P	S, P	S, P	S, P	S, P	S, P	S, P	
6 (Stella)	S, P	S, P	S, P	S, P	S, P	S, P	S, P	S, P	S, P	
7 (Filio)	S, C	S, C	S, C	S, C	S, C	S, C	S, C	S, C	S, C	Conceptual

Table 4

Students' Performance (Success, Failure) and Type of Strategy Used (Conceptual, Procedural, Or Conceptual-Procedural) in the Tasks of Category D

Student	Q.1.23	Q.1.24	Q.1.25	Q.1.26	Q.1.27	Q.1.28	Q.1.29	Q.1.30	Profile
1	F, C	F, C	S, C/P	F, C	S, C	S, C/P	F, C	S, C/P	Conceptual/Procedural
2	S, C/P	S, C/P	S, C	S, C	S, C	F, C	S, C	S, C/P	
3 (Kosmas)	S, C/P	S, C/P	S, C/P	S, C	S, C/P	S, C/P	S, C	S, C/P	
4	F, P	F, P	F, P	F, C	F, P	F, P	F, C	F, C	Procedural
5	F, P	F, P	F, P	F, C	F, P	F, P	F, C	F, C	
6 (Stella)	F, P	F, P	F, C	F, C	F, P	F, P	F, C	F, C	
7 (Filio)	S, C	S, C	S, C/P	S, C	S, C	F, C/P	F, C	S, C	Conceptual

3.2 Procedural Profile

As shown in Table 1, the students of this profile performed very well in the tasks of Category A (Table 1). On the contrary, their performance was very law in the tasks of Category B (Table 2). In particular, Student 3 (hereafter, Stella) failed in all the tasks of this category. She stated that "the nominator shows how many pieces to take" to justify her answer in Q.1.6, and she drew a circle and partitioned it in three unequal parts in Q.1.8 (Figure 1). None of these students exhibited any understanding of the fundamental principle that the fractional parts of the unit should be equal, as also evidenced by their performance in Q.1.7 (Table 2).

Figure 1. Stella's response to Q.1.6, Q.1.8: Representations for the Fractions 1/4 and 2/3, respectively.

All three students failed to represent the improper fraction 5/3 (Q.1.9). Figure 2 presents S5 and Stella's attempts to deal with this task. S4 gave no answer to the problem.

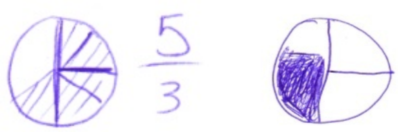

Figure 2. Procedural Profile: Student 5 and Stella's' Attempt to Represent the Fraction 5/3.

In addition, all three students failed in Q.1.10, explaining that the denominator shows how many pieces the pizza had, and the nominator how many pieces were eaten. They also failed in Q.1.11, since they did not consider that the units of reference might be different. Moreover, they all insisted on executing the calculations in Q.1.12 and Q.1.13. When they were explicitly instructed not to do it, they came up with the rule "multiplication makes bigger, whereas division makes smaller".

All students of this profile were flawless in the tasks of Category C, using only procedural strategies. They were, however, very reluctant to try without using paper and pencil, when they were asked to. In case they tried, their responses reflected severe lack of understanding. For example, Stella claimed that 123/220 is greater than 6/5 because the numbers 123 and 220 are greater than 6 and 5, respectively.

The students of this profile failed in all tasks of Category D (Table 4). Again, they relied heavily on procedural strategies, in particular transformation strategies. For example, they all converted fractions into decimals in Q.1.23 and Q.1.24. They also attempted to use this strategy or to perform the calculation in the estimation task Q.1.26, although they were specifically asked not to. Stella, in particular, explicitly stated that it is impossible to solve the task without converting to similar fractions or to decimals first.

Students 4 and 5 applied the instructed method Q.1.25. However, they were not able to interpret the result. Consider, for example, the answer and the explanation provided by Student 5: "George' s milk tastes sweeter, because his proportion 600/100=6 is better than Stella's 200/50=4". On the other hand, Stella's answer indicated that she neglected the multiplicative relations defining the relative quantities that are involved in the situation: "The girl's quantities are rather small compared to the boy's. So I believe that her milk tasted sweeter".

These students' responses to the tasks on dense ordering (Q.1.29, Q.1.30) were immediate and reflected the idea that fractions (or decimals, in case they had converted them) are discrete, like the natural numbers. Stella stated that "there are no other numbers between 2/5 and 3/5, because 3 comes right after 2". According to Stella, one was the smallest positive number, while Students 4 and 5 proposed 0.1.

3.3 Conceptual Profile

As mentioned above, there was only one student placed in this profile, namely Filio. As shown in Table 1, Filio failed in all tasks of Category A, except for Q.1.5, since she was quite competent with equivalent fractions (see also her solution in Q1.25 below). On the contrary, she succeeded in all tasks of Category B (Table 2). She was able to explain adequately her responses. For example, to explain her disagreement with Maria in Q.1.10, Filio said that "I don't know how many pieces this pizza had. Kostas could have eaten 3 pieces, only if the pizza was cut in four". Similarly, in Q.1.11, she exclaimed: "Where are the pizzas? I need to see the pizzas. Are they the same or not?" While dealing with Q.1.12 and Q.1.13, she explicitly stated that the outcome is not necessarily bigger, just because there is multiplication involved. She tried with several numbers, and eventually came up with a generalization: "when we multiply a number a by a fraction smaller than the unit, the product is smaller than the number a".

Filio succeeded in all tasks of Category C (Table 3) using consistently only conceptual strategies. Interestingly, she also succeeded in most of the tasks of Category D (Table 4). Her responses in Q.1.23, Q.1.24, were based on estimation of the fraction magnitudes and a rough approximation of their location on the numbers line. Unlike the students of the Conceptual-Procedural Profile, she didn't attempt to find the exact locations by partitioning the line segments. Quite similar to these students, however, she focused on the relations between the quantities in Q.1.25, employed a transformation strategy, and came up with a solution that is not taught at school: "The 50gr of chocolate powder that Stella put in 200gr milk is half the quantity that George put in 600gr. So I double the quantities 50/200 and I get 100/400. Then, 100 in 400 means more chocolate powder in the milk than 100 in 600! So, Stella's milk tastes sweeter."

Similarly to Kosmas, Filio explicitly stated that there are infinitely many pairs whose product is 3 (Q.1.27). Moreover, she also stated that there are infinitely many numbers between 2/5 and 3/5 (Q.1.30).

Unlike all other participants, she justified her answer using spontaneously a rather sophisticated representation: “If we locate them on the number-line, there is definitely a gap in between. In this gap, there are infinitely many numbers”.

We note that Filio explicitly expressed her discomfort with tasks in which she could not avoid using procedures (e.g., Category A tasks, Q.1.28). We also note that Filio was monitoring her performance during the solution process. She explicitly expressed doubt about responses that were actually incorrect; she also revised certain answers herself. For example, when solving Q.1.18, she initially answered that the fractions 3/4 and 6/7 are equal, because for both one fractional unit is needed to complete the unit. She revised this answer after locating the two fractions on the number line.

3.4 Conclusions

The first phase of the study revealed three different student profiles: The Conceptual-Procedural Profile consisted of three students with quite strong conceptual knowledge of fractions, combined with procedural fluency. These students appeared to prefer conceptual strategies over procedural strategies, when this was possible. One of these students, namely Kosmas (Student 3), was exceptionally strong: not only did he succeed in all tasks, but he also gave the most complete and elaborated answers.

The Procedural Profile consisted of three students who were capable of applying instructed procedures. This capability allowed them to deal very successfully with the tasks that could actually be solved by an instructed procedure. However, these students failed in most tasks that required conceptual knowledge, exhibiting lack of understanding for even the most fundamental fraction ideas. Stella, in particular, failed in the simplest conceptual tasks. These students relied heavily on procedural strategies and avoided consistently to try otherwise. When they did try, they typically failed.

Finally, the Conceptual Profile consisted of one student, namely Filio (Student 7). Filio consistently avoided applying procedures throughout the interview, and she failed when she had to do it. She nevertheless exhibited a firm understanding of fundamental fraction ideas; and thus she managed to deal quite successfully with many tasks by applying consistently conceptual strategies.

Thus, in line with recent discussions regarding the relation between conceptual and procedural knowledge of fractions (e.g., Hallett et al., 2010, 2012), we found individual differences in the way that students combine the two kinds of knowledge. Moreover, we showed that these differences can be extreme – consider, for example, Stella and Filio.

4. Results of the 2nd phase of the study

Table 5 presents the categories that describe the Deep Learning Approach and the Superficial Learning Approach to mathematics, and their indicators. In the following we present excerpts from transcribed interviews of Kosmas, Filio, and Stella, in order to highlight the similarities and the differences in their learning approaches to mathematics, along these categories.

4.1 Goals

Kosmas and Filio repeatedly referred to the importance of learning with understanding in mathematics, which they both juxtaposed with rote learning. For them, learning with understanding meant personal making of meaning. This point is illustrated in the following excerpts, in which Kosmas and Filio explain how they would help a hypothetical younger student that is challenged by the comparison of fractions:

"Perhaps I could try to explain fractions as I understand them. He has to find a personal way of thinking though. He could study the rules. In fact, there are two ways: In the case of fractions, the first one is to memorize the rules and apply them. For example, between two fractions with the same numerator 3/5 and 3/7 the bigger is this one with the smaller denominator. Alternatively, he would compare the two fractions to the unit, that is, notice that 3/7 is closer to 1 than 3/5. There is a difference: In the second case you have understood exactly what happens with fractions-the first is rote learning. You can reach a conclusion regarding which of two fractions is bigger but you don't understand why. Personally, if I saw these two fractions, I would compare the fractions to the unit so as to check the validity of the rule." (Kosmas, Q.2.11)

"I would help him understand the concept of fraction. But, you know, everyone has their own way of thinking. Mathematics is not rote learning, you have to put your mind to the work. […] I could explain to him how to compare fractions based on the rules, but if he wants be really able to compare fractions, I think that he should understand the concept of fraction. He must understand what fractions are and then he will do well in fractions." (Filio, Q.2.11)

Consider also the following excerpts:

"The most important thing is to understand. Knowing the rules will also help you, there is no doubt about it. But understanding is the most important thing." (Kosmas, Q.2.11)

"If I understand the meaning of what I do, then I can solve the exercises." (Filio, Q.2.2)

Table 5

Deep vs. Superficial Learning Approach to Mathematics: Categories and Indicators

Categories	Sub-Categories	Indicators Deep Approach	 Superficial Approach
Goals		Understanding / Personal making of meaning	Focus on what is required /assessed at school
Study Strategies		Combining theory and practice	Memorizing and Rehearsing
		Systematic, long-term time investment	More rehearsing
Awareness of	Understanding	High	Low
	Effectiveness of Own Study Strategies	High	Low
Engagement factors	Task/Strategy Preferences	Conceptual	Procedural
	Motivation	Intellectual challenge	Coping

On the contrary, Stella repeatedly referred, explicitly or implicitly, to the importance of complying with what is assessed at school and appeared to focus exclusively on the material taught at school. This is summarized nicely in the following excerpt:

"What I would advise a younger student is to look at the exercises solved at school, to focus on what is likely to be asked in the exams, and to pay attention to what the teacher has emphasized on." (Stella, Q.2.1)

4.2 Study Strategies

Kosmas and Filio both stressed that in order to study efficiently in mathematics one needs to combine studying theory in depth and extensive practice with exercises. They also expressed their conviction that solving unfamiliar problems is important as a study strategy as well as an indicator of understanding.

> *"You have to know the theory very well so as to understand mathematics. If you only solve exercises, your competence is very limited. One has to understand the theory in depth before trying to solve exercises." (Kosmas, Q.2.2)*
>
> *"If you give me any problem and I can solve it, then it means I have understood well." (Kosmas, Q.2.9)*
>
> *"One should understand the theory very well and practice a lot as well; and solve exercises beyond the ones in the textbook." (Filio, Q.2.2)*

In contrast, Stella's study strategies were limited to memorizing and rehearsing:

> *"Studying what is needed for solving the exercises is pretty much sufficient." (Stella, Q.2.2)*
>
> *"Studying the theory is good, because you have to know some theory to be able to solve the exercises. But I think that it is better to focus on exercises. Personally, I look at what we have done at school, so as to remember how the exercises are solved. I solve them again and again, and then I check if they are correct." (Stella, Q.2.3)*

In addition, unlike Stella, Kosmas and Filio appeared to value the hypothetical students' study strategies in Q.2.3, although they both admitted that they don't study like this.

> *"There is no doubt that this is the appropriate way of studying the theory. […] This is how I should study but, unfortunately, I don't. That's why I am not strong in mathematics." (Kosmas, Q.2.3)*
>
> *"What she does is just fine. I don't study like this, but I wish I did." (Filio, Q.2.3)*

Moreover, Kosmas and Filio referred to the importance of investing time on mathematics studying. They distinguished between merely spending time on studying, and studying systematically and in depth.

> *"Mathematics is a course that has to do a lot with understanding, so you have to study a lot. You have to start systematically in mathematics from the beginning. Gaps are difficult to cover, one needs to dedicate lots of time for both theory and exercises." (Kosmas, Q.2.2)*
>
> *"I was preparing for a mathematics test and I spent lots of time, but only during the last two days before the exam. I believe that studying in depth results to success. If you study superficially, you are not prepared appropriately. When we talk about mathematics, you can't prepare at the last minute. If you do it, you will fail. It is impossible to learn mathematics two days before the exams." (Kosmas, Q.2.4)*
>
> *"It's not only the time spent on studying, it's also the way you study. […] You may feel well-prepared for a test because you have spent lots of time on solving exercises and fail in the end. For example, what has happened to me is to face unfamiliar problems in a test and fail. In that test, our teacher tested whether we can think for ourselves, so he examined us in different tasks than the ones we had solved in the classroom. […] In order to succeed, you must have understood the concepts and have practiced a lot." (Filio, Q.2.4)*

Stella also mentioned time as an important factor of success in mathematics. For Stella, however, spending more time on studying meant more rehearsing:

> *"[One of my classmates] is a very good at math. I believe that I am good too, but not exactly at the same level. […] I think he spends more hours studying than I do. […] Perhaps he solves the exercises more times than I do." (Stella, Q.2.5)*

4.3 Awareness

4.3.1 Awareness of understanding

Kosmas felt confident that he was able to assess his performance in mathematical tasks in general. In fact, he was very accurate in assessing his performance regarding the fraction tasks.

As already mentioned, Filio was monitoring her performance in the fraction tasks and corrected several mistakes herself in the process. She also detected practically all the tasks that she had answered incorrectly. In addition, she was aware that she lacked procedural fluency:

> *"I don't remember rules and procedures regarding fractions. However, if someone reminded me of them, I could apply them." (Filio, Q.2.9)*

Filio acknowledged that fractions require "a lot of thinking" and recalled that she was challenged by fractions at the elementary school. Interestingly, she mentioned that she managed to grasp the meaning of fractions, by connecting the "school fractions" with the fractions she met at her music courses. (Filio, Q.2.6)

Stella, on the other hand, was confident that she had answered pretty much all fraction tasks correctly. She appeared to detect her mistake in Q.1.9, and she revised her answer. However, her second attempt was again incorrect, since it was based on the assumption that 5/3 is "a bit bigger than 0.5". Nevertheless, Stella believed that she had a firm understanding of fractions in general:

> *"I believe that I understand everything about fractions. I never had any difficulty with fractions. I found them very easy at the elementary school, too. In general, I have never had any problems with mathematics, as far as I can remember." (Stella, Q.2.8)*

4.3.2 Awareness of the effectiveness of own study strategies

As mentioned before, Kosmas and Filio both admitted that they did not follow effective study strategies in mathematics, although they recognized and appreciated them. In addition, they both attributed the fact that they didn't excel in mathematics to their own way of studying.

> *"[One my classmates] is really strong in mathematics. I am at a considerably lower level. This is because I don't invest enough time to study seriously in mathematics. [...] Often I only solve the exercises that I have as homework and stop there. [...] I could be as strong as my fellow student, provided that I would be determined to study seriously (Kosmas, Q.2.5)*
>
> *"I could be as good as him [my fellow student]. How? The old-fashioned way: Putting time and effort in studying as I should." (Filio, Q.2.5)*
>
> *On the contrary, not once did Stella question her study strategies:*
>
> *"Every time something went wrong, this happened because I was not so careful. […] Or I thought I knew the material and that there was no need to look at it again, but in fact I did not remember it well. But in cases that I had studied as I should, I believe that stress was responsible for my failure." (Stella, Q.2.4)*

4.4 Engagement Factors

4.4.1 Task/Strategy Preferences

As already mentioned, during the first phase of the study it was more than obvious that Filio resented the tasks that she perceived as procedural. For instance, she grew impatient with Q.1.28 and quitted trying, exclaiming "I've had enough! I spent too much time on this already. I can't do it, I won't do it!".

Kosmas, on the other hand, never expressed any discomfort when he had or chose to apply procedural strategies. In spite of this important difference, these students both expressed their preference for conceptual over procedural tasks, when they were explicitly asked to chose:

> *"This is an easy choice! I would choose the second one, because I do not like using methods. I do know, however, that the first one is easier. At any moment you can open your book and remember how it is solved." (Kosmas, Q.2.10)*

> *"Not the first one, for sure. It's better to think something new, instead of constantly doing the same. I find no meaning in the application or rules and procedures. It is not interesting. It is like rote learning, you know, a method to solve exercises." (Filio, Q.2.10)*

Unlike Kosmas and Filio, Stella showed a clear preference for procedural strategies during the first phase of the studies; and she explicitly stated that she would prefer the standard, procedural task in Q.2.10.

4.4.2 Motivation

As it may be evident by their responses to Q.2.10, Kosmas and Filio were motivated by novel and challenging tasks. There were clear such indications about Kosmas already in the first phase of the study. For instance, when he first saw Q.1.29, his immediate reaction was the following: "The smallest positive number! This is a nice question, isn't it?" In fact, Kosmas was the only participant who chose to deal with the most demanding and unfamiliar tasks first. When asked why, he replied:

> *"I like challenging tasks much more. I find no interest in solving exercises similar to the ones I have met before. The point is to think of something new."*

Similarly, Filio explained her choice of the unfamiliar task in Q.2.10 as follows:

> *"When you try to solve an exercise and you finally discover that something that you thought for yourself is correct, you get a very nice feeling." (Filio, Q.2.10).*

On the contrary, Stella's main concern was to stay on the safe side. As may be evident by her responses presented above, she was mainly interested in good school performance. When she explained why she would prefer the "standard", procedural, task in Q.2.10, she indicated that she was minding the possible failure that guided her choice:

> *"I would choose the first one because it involves operations, which I already know. So I would be sure that I can respond correctly. The second one may involve something I don't know or never met before."*

Finally, we note that for Kosmas and Filio learning with understanding, besides being an important goal in mathematics learning, also had a motivational aspect. Consider, for example, the following excerpts:

> *"If you are to study mathematics, you should understand what you're doing. You should find meaning in what you do." (Filio, Q.2.2)*

> *"[My classmate who excels in mathematics] has a special interest in math, he loves it. He finds meaning in what he does. That's why he dedicates so many hours to studying." (Filio, Q.2.5)*

Both students mentioned that they felt they understood mathematics at the elementary level, but not so at the secondary level. This was due to the fact that procedures are over-emphasized at the secondary level and this appeared to be demotivating for them.

> *"Instruction on fractions is based on algorithms and students do not understand the concept of fraction. For example, in the addition of fractions we learn a priori that fractions must have the same denominator without understanding why. Something similar happens to mathematics teaching in general. We should understand mathematics deeper and I think that teachers must help us. How? I don't know." (Kosmas, Q.2.7).*

4.5 Conclusions

As evidenced by their interview excerpts, Kosmas and Filio exhibited similar features along the categories Goals, (Study) Strategy use, Awareness, and Engagement Factors. Specifically, they both appeared to value understanding and personal making of meaning in mathematics learning; they were convinced that the study of mathematics requires combining deep understanding of theory as well as extensive practice; systematic and long-term time investment was a key issue for them, as they appeared aware that merely spending time on mathematics studying is not enough to succeed in mathematics. Kosmas and Filio showed high awareness of understanding in the domain of fractions; they were also highly aware of their limitations as students in mathematics. Finally, they showed a clear preference for tasks that require conceptual understanding and present an intellectual challenge, which appeared to be motivating for them.

On the contrary, Filio differed across all categories. Specifically, Filio's goal was to cope successfully with what was required at school; her study strategies were limited to memorizing rules and procedures as well as solving similar or even the same exercises repeatedly; she preferred procedural tasks because she was confident that she would succeed. Finally, she showed practically no awareness of her (extremely limited) conceptual understanding of fractions, and no awareness of the limitations of her study strategies.

5. Discussion

Our results support the hypothesis that there are individual differences in the way that students develop conceptual and procedural knowledge of fractions. Similarly to Hallett et al. (2010, 2012), we identified students who were strong with respect to one type of knowledge, but weak with respect to the other. Although the findings of Hallet et al. (2012) indicate that such individual differences become less salient with age, we showed that for some students they remain extreme, even at Grade 9. Consider, for example, Stella and Filio: It appears that for these students conceptual and procedural knowledge of fractions have not developed in a hand-over-hand process, as predicted by the iterative model (Rittle-Johnson et al., 2001).

In addition, our study provides preliminary evidence indicating that the individual student's learning approach to mathematics is worth investigating in relation to individual differences in conceptual and procedural knowledge. Similarly to Stathopoulou and Vosniadou (2007), we found that Kosmas and Filio, who exhibited strong conceptual knowledge of fractions, both valued a deep approach to mathematics learning; whereas Stella, who exhibited poor conceptual knowledge of fractions, appeared to follow a superficial approach. This finding cannot, of course, be generalized, given that it comes from a qualitative study, with small sample. Moreover, it is based on "extreme" cases of individuals. Nevertheless, this qualitative evidence can inform the hypotheses and the design of future quantitative studies.

Investigating individual differences in conceptual and procedural knowledge is important for understanding mathematical development (Canobi, 2004; Hallett et al., 2010, 2012). From an educational perspective, however, encouraging the symmetrical development of the two kinds of knowledge is an important goal, since they are both considered essential for students' mathematical competence (Rittle-Johnson & Schneider, in press). To this end, probably the first step would be to foster learning environments in which both conceptual and procedural knowledge are valued – and also assessed.

Keypoints

- There are individual differences, even extreme, in the way students combine conceptual and procedural knowledge of fractions.
- The individual student's learning approach to mathematics is a factor worth investigating with respect to individual differences in conceptual and procedural fraction knowledge.

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